2. Characteristics of Synchrotron Radiation

2.1 Introduction

The radiation in general is characterized by the following terms: spectral range, photon flux, photon flux density, brilliance, and the polarization. The photon flux is the overall flux collected by an experiment and reaching the sample, the photon flux density is the flux per area at the sample and the brilliance is the flux per area and opening angle of the source. In the following chapter the formulas for the calculation of these terms of the synchrotron radiation emitted from a stored beam in the bending magnet, wiggler and undulator are compiled.

Many authors have established the theory of synchrotron radiation. Today most of the calculations are using the results of the Schwinger theory. According to this theory the shape and intensity within the radiation cone emitted by a radial accelerated relativistic electron beam is given by:

$$\frac{d^2\Phi}{d\theta d\psi} = \frac{3\alpha}{4\pi^2}\gamma^2 \frac{\Delta\omega}{\omega}\frac{I}{e} y^2(1+X^2)\left[K_{2/3}^2\ (\xi) + \frac{X^2}{1+X^2}K_{1/3}^2(\xi)\right] \qquad (2.1)$$

where:

- Φ = photon flux (number of photons per second)
- Θ = observation angle in the horizontal plane
- Ψ = observation angle in the vertical plane
- α = fine structure constant = (1/137)
- γ = electron energy / $m_e c^2$ (me = electron mass, c= velocity of light)
- ω = angular frequency of photons ($\hbar\omega$ = photon energy = ε)
- I = beam current
- e = electron charge = $1.601*10^{-19}$ coulomb
- y = $\omega/\omega_c = \varepsilon/\varepsilon_c$ (ω_c = critical frequency = $3\gamma^3 c/2\rho$
- ε_c = critical photon energy (= $3hc\gamma^3/2\rho$)
- ρ = radius of instantaneous curvature of electron trajectory = E/ecB
  in practical units, $\rho(m) = 3.3356*(E/GeV)/(B/T)$
- c = speed of light (= $2.9979*10^8$ m/s)
- E = electron beam energy
- B = magnetic field strength
- ε_c = $h\omega_c$ [$\varepsilon_c(KeV) = 0.665*(E/GeV)^2*(B/T)$]
- X = $\gamma\psi$ (normalized angle in the vertical plane)
- ξ = $y(1+X^2)^{3/2}/2$

The subscripted K's are modified Bessel functions of the second kind. Equation (2.1) is the basic formula for the calculation of the of the characteristics of the synchrotron radiation. The polarization is given by the two terms within the square brackets.

2.2 Radiation from a Bending Magnet

The photon flux of the synchrotron radiation from the bending magnet is given by the integration of Equation (2.1) over the whole vertical angle. In the horizontal plane the emitted cone is constant and therefore the photon flux is proportional to the accepted angle θ in the horizontal plane:

$$\frac{d\Phi(y)}{d\theta} = 2.458\cdot 10^{13}\frac{Photons}{s\cdot 0.1\% BW\cdot mrad}\cdot(E/GeV)\cdot(I/A)\cdot(\theta/mrad)\cdot G_1(\frac{\varepsilon}{\varepsilon_c}) \qquad (2.2)$$

According to Equation (2.2), the photon flux is proportional to the beam current, the energy and the normalized function $G_1(\varepsilon/\varepsilon_c)$ which depends only from the critical photon energy. This function is given in Figure (2.1) and is illustrating that the spectrum of the synchrotron radiation flux is a continuous one with a maximum at 1/3 of the critical photon energy.

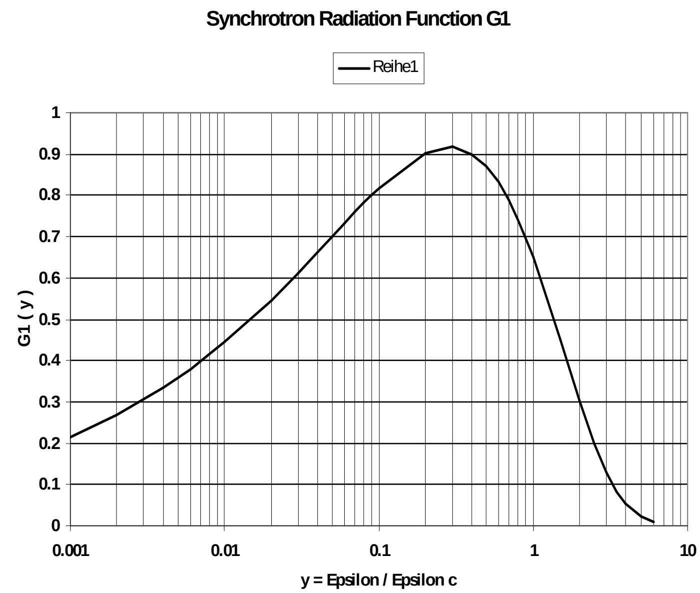

Figure 2.1: The normalized synchrotron radiation function G_1. The flux of the synchrotron radiation from the bending magnet is proportional to this function.

The fluxes emitted from a stored beam in a 1 GeV/1.87Tesla and 2 GeV/1.35Tesla storage ring are presented in Figure (3.3). According to the higher energy, the flux emitted from a 2 GeV machine is a factor of 2 higher. The critical photon energies of both machines are ε_c= 1.24 (1GeV/1.87T) and 3.59 KeV (2GeV/1.35T). The spectrum for the 2 GeV storage ring is roughly one order of magnitude broader, although the critical photon energy of the machines differ only by a factor 3.

The intensity of the synchrotron radiation in the middle of the radiation cone ($\theta = 0$ and $\psi = 0$) is given by the following formula (central intensity):

$$\frac{d^2\Phi(y)}{d\theta d\psi} = 1.326 \cdot 10^{13} \frac{Photons}{s \cdot 0.1\% \cdot mrad\theta \cdot mrad\psi} \cdot (E/Gev)^2 \cdot (I/A) \cdot H_2(\frac{\varepsilon}{\varepsilon_c}) \qquad (2.3)$$

Because the radiation cone is getting narrower with higher energy, the central intensity is proportional the square of the energy. The spectral dependency is given by the normalized synchrotron function $H_2(y) = H_2(\varepsilon/\varepsilon_c)$. This function is presented in Figure (2.2). It is also continuous and has a maximum near the critical photon energy.

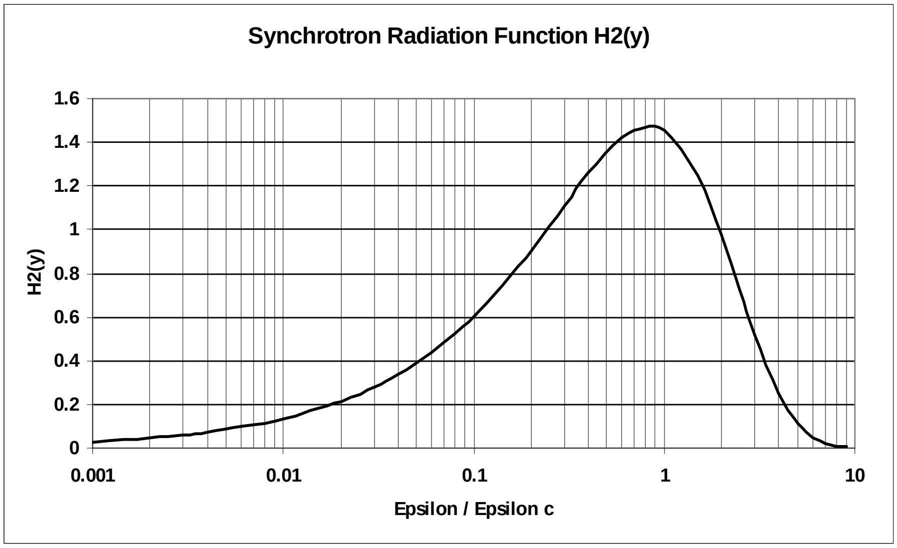

Figure 2.2: The normalized synchrotron radiation function H_2. The brilliance of the synchrotron radiation from the bending magnet is proportional to this function.

From the definition of the flux (Equation (2.2)) and the central intensity (Equation (2.3)) the vertical opening angle of the synchrotron radiation is given by:

$$\sigma_\psi = \frac{1}{\sqrt{2\pi}} \cdot \frac{\langle \frac{d\Phi}{d\theta} \rangle}{\langle \frac{d^2\Phi}{d\theta d\psi} \rangle}(\psi = 0) = \sqrt{\frac{2\pi}{3}} \cdot \frac{1}{\gamma} \cdot \frac{G_1(y)}{H_2(y)} = 0.7395 mrad \cdot \frac{1}{E/GeV} \cdot \frac{G_1(y)}{H_2(y)} \qquad (2.4)$$

The opening angle of the synchrotron radiation for a 2 GeV electron beam is presented in Figure (2.3):

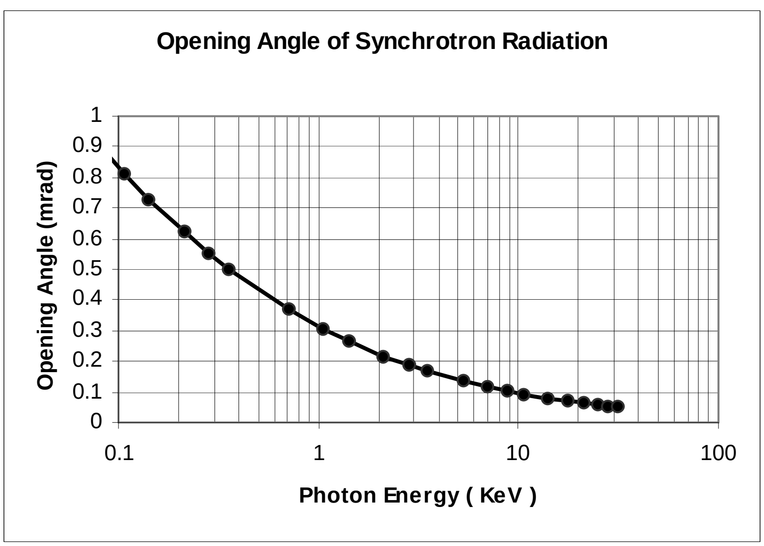

Figure 2.3: Opening angle of the synchrotron radiation emitted from a 2 GeV electron beam

The opening angle of the radiation from 1 and 2 GeV machines are given in Figure (3.4).

The opening angle at the critical photon energy (y=1 or $\varepsilon = \varepsilon_c$) is, according to Equation (2.4):

$$\sigma_\psi (y=1) = 0.331 mrad \cdot \frac{1}{(E/GeV)} \qquad (2.5)$$

For a 2 GeV machine the corresponding angle is 0.166 mrad.

The brilliance of the synchrotron radiation from a bending magnet is given by the central intensity divided by the cross section of the beam:

$$Br = \frac{\langle \frac{d^2\Phi}{d\theta d\psi} \rangle (\psi = 0)}{2\pi \sum_x \sum_y} \qquad (2.6)$$

where

$$\sum_x = \left[\varepsilon_x \beta_x + \eta_x^2 \sigma_E^2 + \sigma_r^2\right]^{1/2}, \qquad \sum_y = \left[\varepsilon_y \beta_y + \sigma_r^2 + \frac{\varepsilon_y^2 + \varepsilon_y \gamma_y \sigma_r^2}{\sigma_\psi^2}\right]^{1/2} \qquad (2.7)$$

and

ε_x (ε_y)	is the electron beam emittance in the horizontal (vertical) plane,
βx (βy)	is the electron beam beta function in the horizontal (vertical) plane,
η_x	is the dispersion function in the horizontal plane,
σ_E	is the rms value of the relative energy spread,
γ_y	is a Twiss parameter in the vertical plane,
σ_ψ	is the rms value of the radiation opening angle,
$\sigma_r=\lambda/(4\pi\sigma_\psi)$	is the diffraction limited source size,
λ	is the observed photon wavelength

At a photon energy of 10 KeV the corresponding photon wavelength is 0.124 nm and the opening angle is smaller than 0.1 mrad (see Figure (2.3)). Both figures result in a diffraction limited source size of σ_r=1.24 µm. The term $\varepsilon_y/\sigma_\psi$ gives a cross section of 2 µm and $\varepsilon_y\gamma_y/\sigma_\psi^2$ has a value between $1*10^{-3}$ and $4*10^{-3}$. These factors are at least one order of magnitude smaller than the beam cross section σ_x and σ_y, hence the overall cross sections in Equation (2.7) reduces for the storage ring SESAME to:

$$\sum_x = \left[\varepsilon_x \beta_x + \eta_x^2 \sigma_E^2\right]^{1/2}, \qquad \sum_y = \left[\varepsilon_y \beta_y\right]^{1/2} \qquad (2.8)$$

and the brilliance of the synchrotron radiation from the bending magnet of a non diffraction limited light source is given by:

$$Br_{Magnet} = \frac{\langle \frac{d^2\phi}{d\theta d\psi} \rangle (\psi = 0)}{2\pi \cdot \left[\varepsilon_x \beta_x + \eta_x^2 \sigma_E^2\right]^{1/2} \cdot \left[\varepsilon_y \beta_y\right]^{1/2}} = \frac{\langle \frac{d^2\Phi}{d\theta d\psi} \rangle (\psi = 0)}{2\pi \sigma_x \sigma_y} \qquad (2.9)$$

The brilliance of a 1 and 2 GeV beam (400 mA) are given in the Figures (3.7) and (3.8). The critical photon energies of the different versions are: SE_1_I = 1.24 KeV, SE_2_I = 4.00 KeV, SE_3_I = 4.00 KeV and SE_4_I = 3.60 KeV. From both figures it follows that the maximum brilliance is around the critical photon energy. Because of the higher energy the brilliances of the 2 GeV beam are one order of magnitude broader than those for a 1 GeV beam. According to the smaller cross sections (emittances) the brilliances of the versions SE_3_I and SE_4_I are of a factor up to 50 higher than those from the 1 GeV beam (version SE_1_I).

2.3 Radiation from a Wiggler

The wiggler is a special magnet with alternating directions of the magnetic field and the trajectory of an electron beam through a wiggler is like a snake as shown in Figure (2.4), it is a sinusoidal oscillation. The trajectory is determined by the maximum slope X' and by the maximum amplitude X_0. Both expressions are given by Equation (2.10).

$$X_0 = \frac{1}{2\pi}\cdot\frac{K}{\gamma}\cdot\lambda_p = \frac{8.13\cdot 10^{-5}}{(E/GeV)}K\lambda_p,\quad X' = K/\gamma,\quad K = 0.934\cdot(B/T)\cdot(\lambda_p/cm) \qquad (2.10)$$

In the "Green Book" and in this "Proposal" different wigglers are foreseen: In the "Green Book" a 7.5 Tesla super conducting wiggler and within this "proposal" a normal conducting one with a field of 2.25 Tesla. The data of these devices are summarized in the Table (2.1):

Table 2.1: Data's of the wigglers foreseen in the "Green Book" and in this "Proposal"

	B_0	λ_W	N_W	L	K	X_0	X'
Green Book	7.50 T	140 mm	6.5	0.91 m	98.1	1.1 mm	50 mrad
Proposal	2.25 T	80 mm	30	2.4 m	16.8	0.055 mm	4.3 mrad

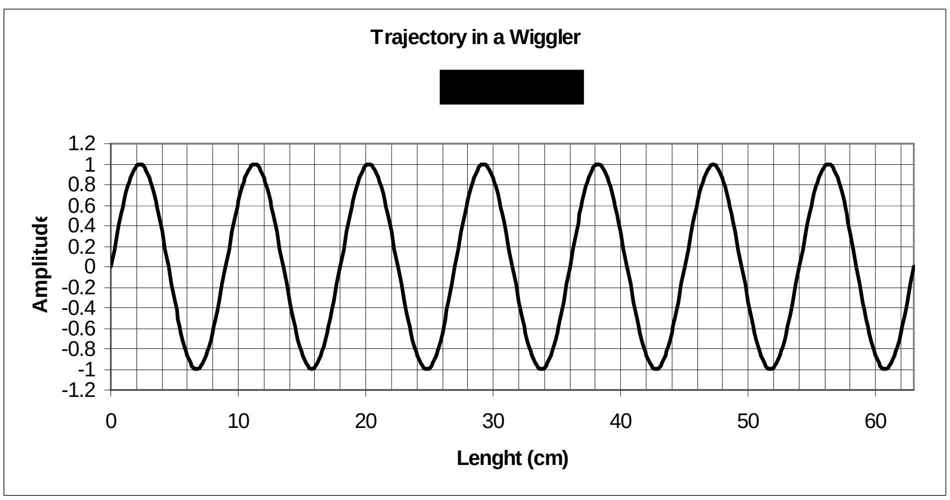

Figure 2.4: Trajectory of an electron beam in a wiggler with a period length of 9 cm

The photon flux as well as the central intensity of the radiation emitted by the wiggler is the same as from the bending magnet but by a factor N_p more intensive, where N_p is the number of poles within the wiggler. The photon flux emitted from the wigglers beams for the "Green Book" and this "Proposal" are presented in the Figures (3.9) and (3.12). Both wigglers have roughly the same critical photon energy (ε_c(Green Book) = 5.0 KeV, ε_c(Proposal) = 6.0 KeV) and therefore the spectrum of the flux is roughly the same.

For the intensity of the photon flux the amplitudes X_0 of the beam oscillations within the wigglers have to be considered (see Table (2.1)). Because of the amplitude of 1.1mm in the "Green Book" design the spot sizes in the wigglers have a difference of 2.2 mm and it is not possible to collect both sources within one beam line. Therefore the useable flux from the wiggler for an experiment is only proportional to half of the number of the poles. For the wiggler in this

"Proposal" it is with an amplitude of 50 μm completely different; here all poles have to be considered. All these arguments are included in the Figures (3.9) and (3.12), with a result, that the flux from the 2 GeV stored beam is of a factor 18 higher than that from the 1 GeV one.

The calculation of the brilliance of wigglers needs to take into account the depth-of-fields, i.e. the contribution to the apparent source size from different poles. The expression for the brilliance of wigglers is:

$$Br_{wi} = \langle \frac{d^2\Phi}{d\Phi d\psi} \rangle (\psi = 0) \cdot \sum \sum_{-N/2}^{N/2} \frac{1}{2\pi} \cdot \frac{\exp\left\{ \frac{-1}{2} \left[\frac{X_0^2}{\sigma_x^2 + z_n^2 \sigma_x'^2} \right] \right\}}{\left\{ \left[\sigma_x^2 + z_n^2 \sigma_x'^2 \right] \cdot \left[\frac{\varepsilon_y^2}{\sigma_\psi^2} + \sigma_y^2 + z_n^2 \sigma_y'^2 \right] \right\}^{1/2}} \qquad (2.11)$$

where:

$$z_n = \lambda_p \left(n + \frac{1}{4} \right) \qquad (2.12)$$

σ_x, σ'_x, σ_y and σ'_y are the rms transverse size and angular divergence of the electron beam at the center of an insertion straight section ($\alpha_x = \alpha_y = 0$). This means that the brilliance of the wiggler, calculated according to Equation (2.11), is normalized to the middle of the straight section.

The exponential factor in Equation (2.11) arises because the wigglers have two points, separated by $2*X_0$ according to Equation (2.10) and indicated in Figure (2.4). (The influence of this two source points upon the photon flux has been discussed before.) The sum in Equation (2.11) goes over all poles of the wigglers. As already discussed under the radiation of the bending magnets the factor ($\varepsilon_y/\sigma_\psi$) is at least a factor 10 smaller than the cross section σ_y and can be neglected. The expression $z_n*\sigma_y'$ is the increase of the source size from the center of the insertion device.

Instead of normalizing the brilliance to the center of the straight section, the cross sections of the beam size $\sigma_x(n)$ and $\sigma_y(n)$ at the position of the different poles can be used, with the result that the expression for the brilliance will be simpler:

$$Br_{Wig} = \langle \frac{d^2\Phi}{d\theta d\psi} \rangle (\psi = 0) \cdot 2 \cdot \sum_{n=1}^{n=N/2} \frac{1}{2\pi} \cdot \frac{\exp\left\{ -\frac{1}{2} \left[\frac{X_0^2}{\sigma_x^2(n)} \right] \right\}}{\sigma_x(n) \cdot \sigma_y(n)} \qquad (2.13)$$

The inverse of the dominator of Equation (2.13) for different beta functions is presented in Figure (2.5). For high beta functions (4 to 22 m/rad) the cross section of the beam doesn't change very much and the brilliance is according to Figure (2.5) proportional to the length of the wiggler. For small beta functions (0.05 and 0.4 m/rad) the brilliance of the wiggler will saturate, because the cross section of the beam in the outer parts of the wiggler gets high and the contribution to the brilliance is small and can therefore be neglected. According to Figure (2.5) the optimized betatron functions should be in the range of 0.6 to 1.0 m/rad and the length should be in the range of 2 to 3 m.

The brilliance of the radiation emitted from the wigglers within the "Green Book" and this "Proposal" are presented in the Figures (3.11) and (3.14). Figure (3.14) is that one with the "mini-beta-sections". Again, because of the same critical photon energies the emitted spectrum covers the same range. However, because of the different cross sections of the beam the intensity is different. For the versions SE_3_1 and SE_4_1 the intensity is roughly the same, but in comparison to version SE_1_2 they have a factor of 40 higher intensity. The version SE_2_1 is of a factor 5 more intensive. The picture changes completely by introducing "mini-beta-sections". The brilliance of the wiggler radiation for this version is presented in Figure (3.14), with the result that the brilliance of the version SE_4_2 is of a factor 400 higher than that from the version SE_1_2.

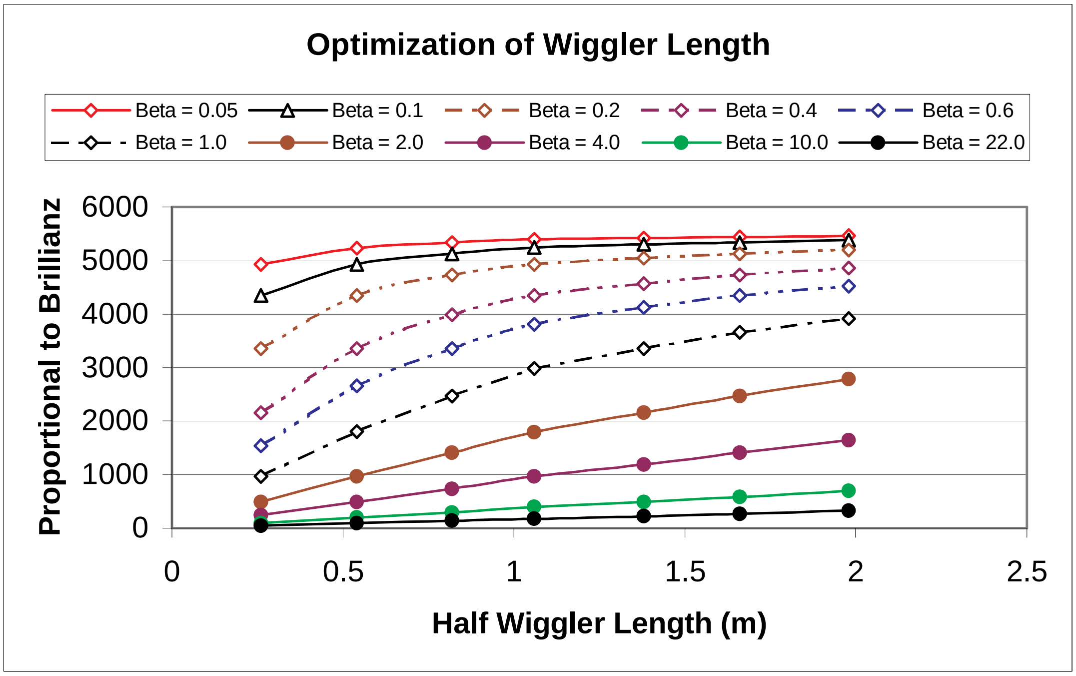

Figure 2.5: Dependency of the brilliance of wiggler radiation upon the beta functions in the middle of the wiggler. The proportionality is given by the sum of all inverse cross sections at the different poles of the wiggler.

In order to optimize the brilliance of the wiggler radiation the beta functions in the middle of the straight sections should be as small as possible (mini-beta-section) and the length of the wiggler has only to be around 2 m, because the outer regions of the wiggler don't have any significant contribution to the brilliance.

2.4 Radiation of an Undulator

The opening angle σ_ψ of the synchrotron radiation from the bending magnet at the critical photon energy ε_c is according to Equation (2.4) roughly 0.655/γ or 1/γ. The maximum slope of the electron trajectory in a wiggler is X' = K/γ. For values of K in the range between 1 to 2 the deflection angle in a wiggler is within the opening angle of the synchrotron radiation. For this special case the radiation from different periods interferences coherently, thus producing sharp peaks with the result of completely different characteristics. This radiation is called undulator radiation and the corresponding insertion devices are undulators.

The undulator emits radiation only at characteristics photon energies:

$$\varepsilon_n = 0.949 KeV \cdot (E/GeV)^2 \cdot \frac{n}{(\lambda_{Und}/cm)\cdot(1+K^2/2)} \qquad (2.14)$$

with the bandwidth:

$$\frac{\Delta\varepsilon_n}{\varepsilon_n} = \frac{1}{nN} \qquad (2.15)$$

where:

n = Harmonic number (n = 1, 3, 5, 7, …..)

N = Number of periods
λ_{Und} = Period length of the undulator
K = Deflection parameter (see Equation (2.10))

The opening angle of the undulator radiation cone is:

$$\sigma_{r'} = \frac{1}{\gamma} \cdot \sqrt{\frac{(1+K^2/2)}{2Nn}} \qquad (2.15)$$

For the illustration of the characteristics of the undulator radiation, the following example shall be used:

E = 2 GeV, λ_{Und} = 40 mm, K = 2 and N = 50

$$\varepsilon_1 = 0.316 KeV \text{ and } \varepsilon_9 = 2.85 KeV$$

$$\Delta\varepsilon_1 = 6.32 eV \text{ and } \Delta\varepsilon_9 = 6.32 eV$$

$$\sigma_{r'}(1) = 0.0443 mrad \text{ and } \sigma_{r'}(9) = 0.0148 mrad$$

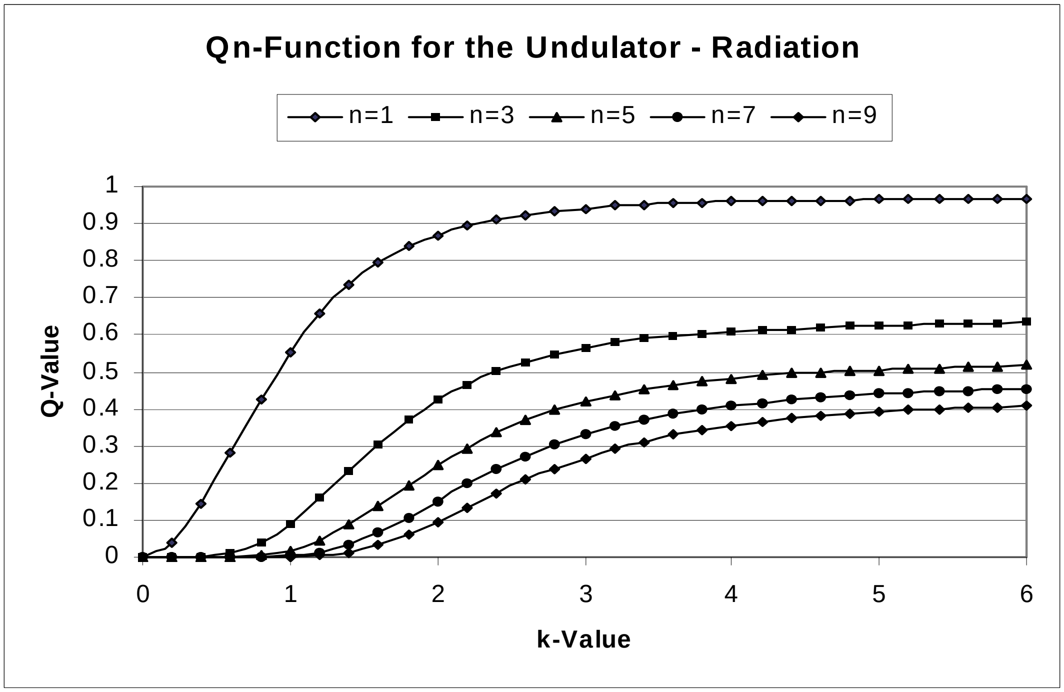

Figure 2.6: The undulator radiation function $Q_n(K)$ for the calculation of the radiation flux

The flux of the undulator radiation within the cone of the harmonics is given by (in practical units):

$$\Phi_{Und}(n,K) = 1.432 \cdot 10^{14} \cdot N_{Und} \cdot (I/A) \cdot Q_n(K) \text{ [Photons/(s 0.1\%BW)]} \qquad (2.16)$$

According to Equation (2.16) the flux of the undulator radiation is proportional to the number of periods, the current and the function $Q_n(K)$. It is independent of the energy of the electrons. The function $Q_n(K)$ is given in Figure (2.6) for the harmonics n =1 to n= 9. To reach a high photon flux (also for higher harmonics), the deflection parameter K should be in the range of 2 to 3.

Table 2.2: Photon energies and fluxes of an undulator installed at SESAME

N	1	3	5	7	9
ε_c [KeV]	0.316	0.948	1.58	2.212	2.88
Φ_{Und} [Pho/s 0.1BW	$2.25*10^{15}$	$1.23*10^{15}$	$7.15*10^{14}$	$4.29*10^{14}$	$2.86*10^{14}$

The fluxes of the different harmonics and the corresponding photon energies are summarized in Table (2.2). In comparison with a wiggler (see Figure (3.9)), only the fluxes of the harmonics 1 and 3 are higher than that from the wiggler. The photon energies of these harmonics are smaller than 1 KeV, hence they can't be used for hard X-ray experiments.

On axis the peak intensity of the nth harmonic of the undulator radiation is given by (in practical units [photons / (s 0.1%BW mr^2)]):

$$\frac{d^2\Phi_{Und}}{d\theta d\psi}(\theta=\psi=0)=1.744\cdot 10^{14}\cdot N_{Und}^2\cdot(E/GeV)^2\cdot(I/A)\cdot F_n(K) \qquad (2.17)$$

The peak intensity is proportional to the electron current, the radiation function $F_n(K)$ and the square of the period number. The opening angle of the radiation cone is inversely proportional to the energy, and therefore the peak intensity according to Equation (2.17) is proportional to the square of the energy.

The radiation function $F_n(K)$ is presented in Figure (2.8). To reach maximum central intensity, the k-value should be in the range of 2 to 3.5 for the higher harmonics.

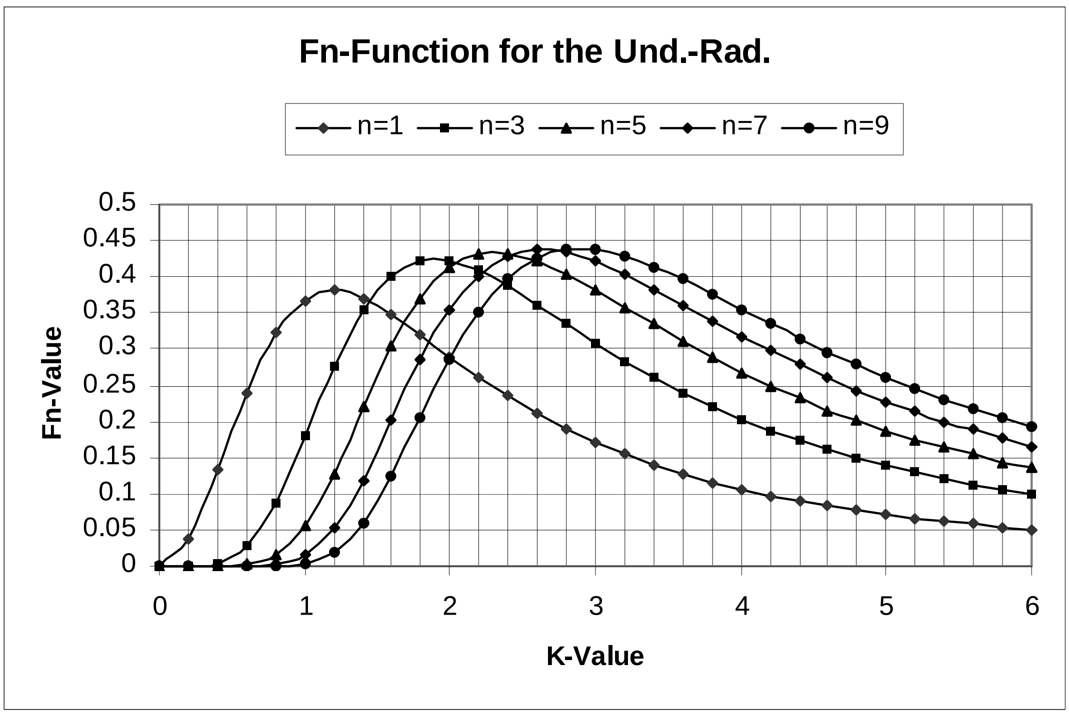

Figure 2.7: The undulator radiation function F_n(K) for the calculation of the central intensity of the undulator radiation.

The brilliance of the undulator radiation is given by:

$$Br_{Und}=\frac{\Phi_{Und}(\theta=\psi=0)}{\left(2\pi\sum_x\sigma\sum_y\sigma\right)\cdot\left(2\pi\sum_x\sigma'\sum_y\sigma'\right)} \qquad (2.18)$$

where

$$\sum\nolimits_x \sigma = \sqrt{\sigma_x^2 + \sigma_r^2}\ ,\ \sum\nolimits_y \sigma = \sqrt{\sigma_y^2 + \sigma_r^2}\ ,\ \sum\nolimits_x \sigma' = \sqrt{\sigma_x'^2 + \sigma_r'^2}\ ,\ \sum\nolimits_y \sigma' = \sqrt{\sigma_y'^2 + \sigma_r'^2} \qquad (2.19)$$

$$\sigma_{x,y} = \sqrt{\varepsilon_{x,y} \cdot \beta_{x,y}}\ ,\ \sigma_{x,y}' = \sqrt{\varepsilon_{x,y} / \beta_{x,y}}\ ,\ \sigma_r = \frac{1}{4\pi}\sqrt{\lambda L}\ ,\ \sigma_r' = \sqrt{\lambda / L} \qquad (2.20)$$

Inserting λ according to Equation (2.14), σ_r and σ'_r are given by:

$$\sigma_r = \frac{\lambda_{Und}}{4\pi\gamma}\sqrt{\frac{(1+K^2/2)\cdot N}{2n}}\ ,\ \sigma_r' = \frac{1}{\gamma}\sqrt{\frac{(1+K^2/2)}{2nN}} \qquad (2.21)$$

For the above mentioned undulator the cross sections σ_r and divergences σ_r' according to Equation (2.21) are summarized in the Table (2.3).

Table 2.3: Cross sections and divergences of the undulator radiation emitted at SESAME

n	1	3	5	7	9
σ_r [µm]	7.05	4.07	3.15	2.66	2.35
σ_r' [µrad]	44.3	25.6	19.8	16.7	14.8

For the electron, the cross sections and divergences are: σ_x = 365 µm, σ_y = 28.7 µm, σ_x' = 46 µrad, σ_y' = 11.6 µrad. With these values the combined cross sections Σx,y and divergences Σx,y' according to Equation (2.19) are summarized in Table (2.4):

Table 2.4: Overall cross sections and divergences for the calculation of the undulator brilliance

N	1	3	5	7	9
Σ_x [mm]	0.365	0.365	0.365	0.365	0.365
Σ_y [mm]	0.0296	0.0290	0.0289	0.0288	0.0288
Σ_x*Σ_y [mm^2]	1.08E-2	1.06E-2	1.05E-2	1.05E-2	1.05E-2
Σ_x' [mrad]	0.0638	0.0526	0.0500	0.0490	0.0483
Σ_y' [mrad]	0.0458	0.0281	0.0229	0.0204	0.0188
Σ_x'*Σ_y' [mrad^2]	2.92E-3	1.48E-3	1.15E-3	1.00E-3	9.08E-4

The brilliance of the different harmonics and the different contributions are given in Table (2.5):

Table 2.5: Parameters for the calculation of the undulator brillinace

n	1	3	5	7	9
ε_n [keV]	0.316	0.948	1.58	2.12	2.88
Φ_{Und} [Phot/s0.1BW]	2.25E+15	1.23E+15	7.15E+14	4.29E+14	2.85E+14
Area*Angle [mm^2*mrad^2]	1.24E-3	6.19E-4	4.77E-4	4.15E-4	3.76E-4
B_r(Und) [Phot./s*mm^2*mra^2*0.1BW]	1.81Ê+18	1.99E+18	1.50E+18	1.03E+18	7.58E+17

The brilliance of the undulator is for the 1st harmonics (0.316 KeV) two orders and for the 9th (2.88 KeV) one order of magnitude higher than that from the wiggler. The scientific case for SESAME has to show if this is interesting for the users.

With the Equations (2.19) to (2.21) the brilliance can be calculated accordingly in practical units [photons/(s 0.1%BW mm^2 mrad^2)]:

$$Br_{Und} = \frac{1}{A \cdot B} \cdot 1.745 \cdot 10^{14} \cdot \left(I / A\right) \cdot N^2 \cdot \left(E / GeV\right)^2 \cdot F_n(K) \qquad (2.22)$$

where:

$$A = \left(2\pi\right) \sum\nolimits_x \sigma \sum\nolimits_y \sigma \text{ and } B = \sqrt{\left(1 + \sigma_x^{'2} / \sigma_r^{'2}\right)} \cdot \sqrt{\left(1 + \sigma_y^{'2} / \sigma_r^{'2}\right)} \qquad (2.23)$$